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Analysis of transverse lamina cracks in multi-directionally 0°, ±45° and 90° oriented reinforced laminates and a damage tolerance design of composite plates *

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Abstract—Analytical approach to transverse lamina cracking of ply-groups in a multi-axially (0°, ±45° and 90° oriented) laminated composite material has been proposed in the present paper. Quasi three-dimensional stress and strain analysis of a ply-group with the through-the-width transverse lamina cracks was carried out by using finite differential total force method, assuming periodical arrangement of cracks along the longitudinal direction. Equivalent elastic moduli and Mode I and II energy release rates of the damaged ply group are evaluated. Effects of crack density, in-plane elastic moduli, ply-group thickness and fiber orientation on the equivalent elastic moduli and energy release rate are expressed as interpolation functions, which are applicable to general multi-axial composite laminates. Simple calculation procedure has been proposed for damage tolerance design of laminated composite structures.

Keywords: Transverse cracks; energy release rate; stiffness reduction; crack density.

1. INTRODUCTION

Fiber reinforced composite materials show excellent mechanical properties in the longitudinal direction, but they have the poor transverse and interlaminar properties. Because of the orthotropic properties, transverse lamina cracking and delamination are the initial and critical failure modes of composite structures. Damage tolerance design, which can deal with the transverse lamina cracking, should be effective in producing structural improvement.

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In order to establish the damage tolerance design for continuous fiber reinforced composite laminate structures, investigation into the mechanism of transverse lamina cracking is an important research issue. Highsmith and Reifsnider [1] have studied transverse lamina cracks based on shear-lag theory. Hashin [2] and McCartney [3] have analyzed the stress and displacement fields of a cross-ply laminate with periodically arranged transverse lamina cracks. Nairn [4] applied Hashin's solutions to thermal residual stress problem. Ogihara *et al.* [5] applied McCartney's solution to failure analysis of cross-ply laminates. Tohgo *et al.* [6] extended Hashin's solution to two-dimensional finite element analysis. The subject of these theoretical studies is limited to the cross-ply laminates.

Transverse lamina cracks in multi-directionally reinforced composite laminates such as quasi-isotropic laminates have been studied by Shahid and Chang [7], Allen *et al.* [8] and by Chatterjee *et al.* [9]. Adolfsson and Gudmundson [10] and Salpekar and O'Brien [11] have proposed finite element models. Finite element solutions give interesting results, but are not applicable to general design problems.

Kageyama *et al.* [12] have proposed a finite difference stress analysis method for transverse lamina cracks in general multi-directionally reinforced laminates. Both stress and energy balance criteria are applied to estimation of failure stress. Effect of ply group thickness on onset and extension of cracks has been evaluated theoretically and experimentally. In the present paper, the numerical method has been improved by considering the resultant forces. The interpolated solutions have been given for general design code as a function of elastic properties, fiber orientation and ply group thickness.

2. ANALYSIS OF TRANSVERSE LAMINA CRACKS USING RESULTANT FORCE METHOD

2.1. Analysis model

As illustrated in Fig. 1, this paper deals with a $[\theta_1/\theta_2/\ldots/\theta_n]$s laminate consists of unidirectionally reinforced laminae with different direction. It is assumed that the

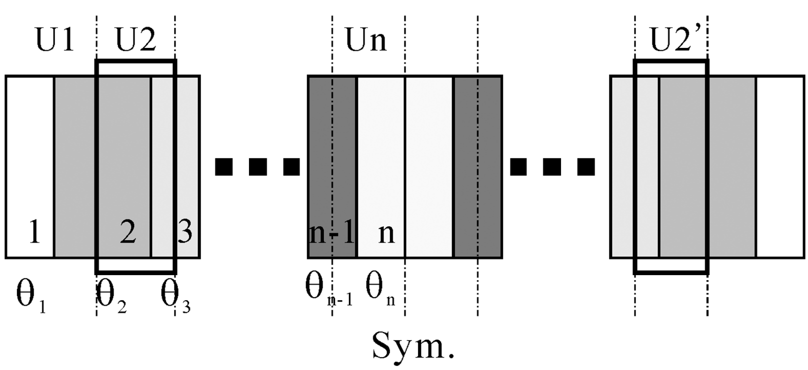

Figure 1. Definition of a unit laminate of a symmetrically laminated composite plate.

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thickness of the θ_i layer (ply group i) is uniform through the laminate, and that the fiber orientation angles of θ_{i-1} and θ_{i+1} layer next to the θ_i layer are different from the fiber orientation angle of θ_i layer. Except for the outside layer (θ_1 layer) and middle layer (θ_n layer), a laminate is divided into $(n - 1)$ units as schematically shown in Fig. 1, where a unit Ui has the middle surfaces of θ_i and θ_{i+1} layers as the boundaries. The left side of the unit Ui in the figure is considered as a $[\theta_i/\theta_{i+1}]$ laminate, and the right side is also considered as a $[\theta_{i+1}/\theta_i]$ laminate, so these are considered as a $[\theta_i/\theta_{i+1}]$s symmetric laminate by combining both laminates.

If we assume that transverse cracks start along with fiber direction in θ_{i+1} layer, the layers can be considered as $[\theta/90]$s laminate where θ is $\theta = \theta_i + \phi = \theta_i - \theta_{i+1} + 90$ by rotating the coordinate system by $\phi = 90 - \theta_{i+1}$ degrees. Then the cracks are considered as transverse cracks in the 90-degree layer in a symmetric laminate. So, if the solution of transverse cracks of 90-degree layer in a $[\theta/90]$s laminate can be found, the behavior of the whole laminate can be analyzed by rotating the coordinate system back and combining each of the units by applying lamination theory.

As illustrated in Fig. 2, this paper deals with unit laminate model with equally spaced perforated transverse lamina cracks in 90-degree layer. As the units are created in the way as described above, the surfaces of both θ layers are the mid-plane of the original layer, so there must be a continuity of displacement in thickness direction. In this paper, it is assumed that the displacement, w, in the thickness direction is fixed as described later.

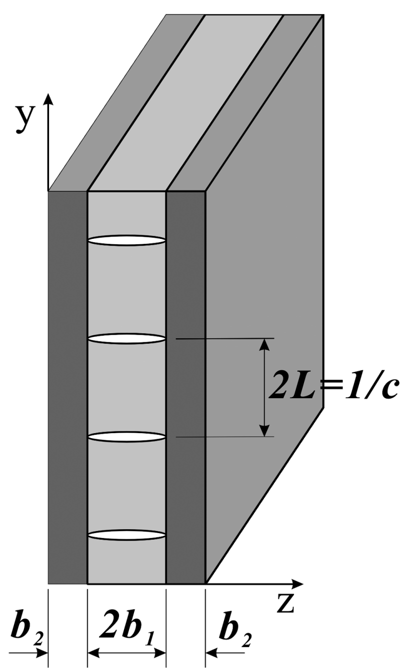

Figure 2. Unit laminate with transverse lamina cracks in 90 deg layer.

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2.2. Assumption of change in stiffness as a whole laminate

Crack density of transverse cracks c is defined as the crack area of unit volume. Assuming that there are transverse cracks at a $2L$ interval in the 90-degree layer in a $[\theta/90]_s$ laminate. The crack density c is $1/(2L)$. The 90-degree layer is called Layer 1, θ layer is called Layer 2, and we assume their thicknesses are b_1 and b_2. The elastic behavior of a laminate with these cracks is discussed here.

When defining longitudinal, transverse and thickness directions as L, T and Z, respectively, change of gross (average) elastic modulus occurs only in E_T and G_{LT} by transverse cracks and it is assumed that the change in E_L and ν_{LT} can be ignored. As for Layer 1 and Layer 2, the stiffness matrix for each layer can be defined, and it is assumed that only $\bar{Q}_{22}^{(1)}(c)$ and $\bar{Q}_{66}^{(1)}(c)$ in stiffness matrix of Layer 1 (90-degree layer) change by the crack density, and the elements of stiffness matrix of Layer 2 do not change. Note that the change in average stiffness of Layer 1 by crack density is effected by neighboring Layer 2. And it is also assumed that the y-directional average stiffness as a laminate can be calculated by lamination theory and can be given in next formula for the whole thickness average.

$$(b_1 + b_2)\bar{Q}_{22}^{(c)}(c) = b_1\bar{Q}_{22}^{(1)}(c) + b_2 Q_{22}^{(2)}, \tag{1}$$

$$(b_1 + b_2)\bar{Q}_{66}^{(c)}(c) = b_1\bar{Q}_{66}^{(1)}(c) + b_2 Q_{66}^{(2)}. \tag{2}$$

Though $[\theta/90]_s$ is a symmetric laminate, the laminate is not balanced except for the $\theta =$ 0-degree case, so there is a cross-elasticity effect. The 90-degree layer does not have these components, so cross-elasticity stiffness components do not change with the crack density. After all, crack density affects only $\bar{Q}_{22}^{(1)}(c)$ and $\bar{Q}_{66}^{(1)}(c)$ in components of stiffness matrix of a laminate.

$$\frac{\partial \bar{Q}_{22}^{(c)}(c)}{\partial c} = \frac{b_1}{b_1 + b_2} \cdot \frac{\partial \bar{Q}_{22}^{(1)}(c)}{\partial c}, \quad \frac{\partial \bar{Q}_{66}^{(c)}(c)}{\partial c} = \frac{b_1}{b_1 + b_2} \cdot \frac{\partial \bar{Q}_{66}^{(1)}(c)}{\partial c}. \tag{3}$$

The effects of crack density on the average stiffness of Layer 1 and a unit laminate are affected by the thickness of neighboring Layer 2, arrangement of fiber directions and their elastic modulus. So each unit must be calculated with specified neighboring layers.

2.3. Governing equation of micro-model

As illustrated in Fig. 3, transverse lamina cracks in $[\theta/90]_s$ laminates are discussed in this paper with the basic model (micro-model) where the interval of cracks is $2L$, average tensile stress in 0-degree direction is σ_{y0} and average in-plane shear stress is τ_{xy0}. y–z coordinate system is defined as illustrated and the assumption of this paper is also applied here. So, displacement, stress and strain do not depend on x, and they are functions of y and z. And as for displacement of thickness direction w, $\partial w/\partial y$ and $\partial w/\partial z$ can be ignored because of the assumption of continuity of displacement in the thickness direction in the original surface. In reference to equation (3),

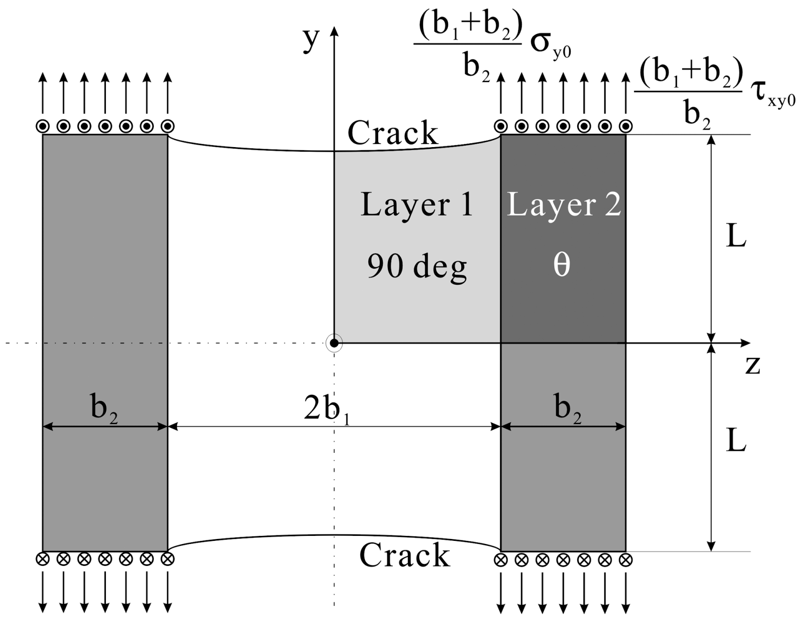

Figure 3. Micro model of analysis.

longitudinal tensile components and in-plane shear components in stiffness matrix can be treated separately.

Only y direction displacement v is discussed and the x direction displacement is assumed to be zero ($u = 0$), when the laminate is subjected to only longitudinal deformation. In this condition, only ε_y and γ_{yz} are not zero in strain components, and defined by using displacement v.

$$\varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{yz} = \frac{\partial v}{\partial z}. \tag{4}$$

All the other strain components are zero. The stress components according to the strain described above are σ_y and τ_{yz}. These stress components must satisfy the equilibrium condition in Layer 1 as defined in equation (5).

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0. \tag{5}$$

The equilibrium condition on the boundary between Layer 1 and Layer 2 is given by equation (6). The out-of-plane shear stress, τ_{yz}, is assumed to be zero.

$$\frac{\partial \sigma_y}{\partial y} - \frac{\tau_{yz}}{b_2} = 0. \tag{6}$$

The constitutive law of linear elasticity of non-zero stress and strain components is given by equation (7).

$$\sigma_y = Q_{22}^{(1)} \cdot \varepsilon_y, \quad \tau_{yz}^{(1)} = Q_{44}^{(1)} \cdot \gamma_{yz}. \tag{7}$$

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In equation (7), stiffness coefficients $Q_{22}^{(1)}$ and $Q_{44}^{(1)}$ are given by the following formulae by using the engineering elastic constants.

$$Q_{22}^{(1)} = \frac{E_{\mathrm{T}}}{1 - \nu_{\mathrm{LT}}\nu_{\mathrm{TL}}}, \quad Q_{44}^{(1)} = G_{\mathrm{TZ}}. \tag{8}$$

The components of out-of-plane shear stress in Layer 2 can be ignored and the following equation is given. $Q_{22}^{(2)}(\theta)$ is tensile stiffness coefficient of unidirectionally reinforced laminate with angle θ.

$$\sigma_y = Q_{22}^{(2)}(\theta) \cdot \varepsilon_y, \quad \tau_{yz} = 0. \tag{9}$$

The compatibility condition of Layer 1 and the boundary surface between Layer 1 and Layer 2 are described as equation (10) from equation (4).

$$\frac{\partial \varepsilon_y}{\partial z} = \frac{\partial \gamma_{yz}}{\partial y}. \tag{10}$$

In-plane shear deformation is analogical with longitudinal deformation when replacing stiffness coefficient, stress and strain components as follows;

$$\boldsymbol{v} \Leftrightarrow \boldsymbol{u}, \quad \boldsymbol{\sigma}_y \Leftrightarrow \boldsymbol{\tau}_{xy}, \quad \boldsymbol{\tau}_{yz} \Leftrightarrow \boldsymbol{\tau}_{zx}, \quad \boldsymbol{\varepsilon}_y \Leftrightarrow \boldsymbol{\gamma}_{xy}, \quad \boldsymbol{\gamma}_{yz} \Leftrightarrow \boldsymbol{\gamma}_{zx},$$
$$\boldsymbol{S}_{22} \Leftrightarrow \boldsymbol{S}_{66}, \quad \boldsymbol{S}_{44} \Leftrightarrow \boldsymbol{S}_{55}.$$

$Q_{66}^{(1)}$ and $Q_{55}^{(1)}$ are given by the engineering elastic constants.

$$Q_{66}^{(1)} = G_{\mathrm{LT}}, \quad Q_{55}^{(1)} = G_{\mathrm{ZL}}. \tag{11}$$

2.4. Differential equations of resultant force

P_y, given in the following incremental equation, is called a resultant force.

$$\mathrm{d}P_y = -\sigma_y \mathrm{d}z + \tau_{yz}\mathrm{d}y. \tag{12}$$

The equation above can be totally differentiable, and P_y is a state quantity or path independent. In consequence, P_y can be described as a function of the location, $(P_y(y, z))$ and the related stresses components can be defined as following.

$$\sigma_y = -\frac{\partial P_y}{\partial z}, \quad \tau_{yz} = \frac{\partial P_y}{\partial y}. \tag{13}$$

The stresses components that are obtained from equation (13) always satisfy the equilibrium condition as given by equation (5). As illustrated in Fig. 4, Layer 1 is divided into M and N units in x and y directions, respectively; and Layer 2 is also divided into N units in the y direction. The intervals of this division in both the z direction (Δz) and the y direction (Δy) are shown in Fig. 4. When defining $P_y(i, j)$ at the nodes of the element, stresses in Layer 1 can be obtained by

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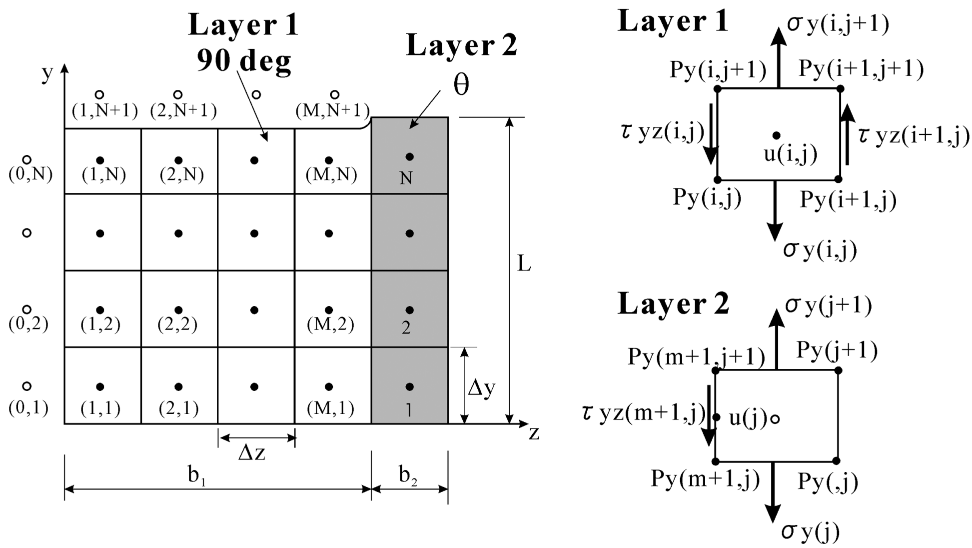

Figure 4. Mesh division of unit model and definition of stresses.

differentiation from the resultant force.

$$\sigma_y(i,\, j) = -\frac{P_y(i+1,\, j) - P_y(i,\, j)}{\Delta z},$$
$$\tau_{yz}(i,\, j) = \frac{P_y(i,\, j+1) - P_y(i,\, j)}{\Delta y}. \tag{14}$$

The longitudinal stress in Layer 2 is also given by equation (15).

$$\sigma_y(j) = -\frac{P_y(j) - P_y(M+1,\, j)}{b_2}. \tag{15}$$

Substituting equations (7) and (13) into equation (10), we can obtain the fundamental differential equations of the resultant force.

$$\frac{1}{Q_{22}^{(1)}\Delta z^2}\big[P_y(i+2,\, j+1) - 2P_y(i+1,\, j+1) + P_y(i,\, j+1)\big]$$
$$+ \frac{1}{Q_{44}^{(1)}\Delta y^2}\big[P_y(i+1,\, j+2) - 2P_y(i+1,\, j+1) + P_y(i+1,\, j)\big] = 0, \tag{16}$$
$$(i = 1, 2, \ldots, M-1 \quad j = 1, 2, \ldots, N-1),$$

$$\frac{1}{Q_{22}^{(1)}\Delta z^2}\big[P_y(i+2, 1) - 2P_y(i+1, 1) + P_y(i, 1)\big]$$
$$+ \frac{2}{Q_{44}^{(1)}\Delta y^2}\big[P_y(i+1, 2) - P_y(i+1, 1)\big] = 0 \tag{17}$$
$$(i = 1, 2, \ldots, M-1).$$

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Compatibility on the boundary between Layer 1 and Layer 2 can be expressed as differential equations (18) and (19).

$$\begin{aligned} &-\frac{2}{Q_{22}^{(2)}(\theta)b_2\Delta z}\big[P_y(j+1) - P_y(M+1, j+1)\big] \\ &\quad + \frac{2}{Q_{22}^{(1)}\Delta z^2}\big[P_y(M+1, j+1) - P_y(M, j+1)\big] \\ &= \frac{1}{Q_{44}^{(1)}\Delta y^2}\big[P_y(M+1, j+2) - 2P_y(M+1, j+1) + P_y(M+1, j)\big], \\ &\quad (j = 1, 2, \ldots, N-1), \end{aligned} \tag{18}$$

$$\begin{aligned} &-\frac{1}{Q_{22}^{(2)}(\theta)b_2\Delta z}\big[P_y(1) - P_y(M+1, 1)\big] + \frac{1}{Q_{22}^{(1)}\Delta z^2}\big[P_y(M+1, 1) - P_y(M, 1)\big] \\ &= \frac{1}{Q_{44}^{(1)}\Delta y^2}\big[P_y(M+1, 2) - P_y(M+1, 1)\big]. \end{aligned} \tag{19}$$

The boundary condition on the y-axis is $\tau_{yz} = 0$, so $P_y(1, j) = 0$ $(j = 1, 2, \ldots, N+1)$ shall be satisfied. The stress-free condition on the crack surface is $\sigma_y = 0$, so $P_y(i, N+1) = 0$ $(i = 1, 2, \ldots, M+1)$ shall be satisfied. On the free surface of Layer 2, $z = b_1 + b_2$, the shear stress, τ_{yz}, shall be zero, and the load condition of applied stress, σ_{y0}, shall be satisfied. The loading conditions are given by equation (20).

$$P_y(j) = P_{y0} = -(b_1 + b_2)\sigma_{y0} \quad (j = 1, 2, \ldots, N+1). \tag{20}$$

Then there are $M * N$ variables ($P_y(i, j)$ $(i = 2, 3, \ldots, M+1,\ j = 1, 2, \ldots, N)$), and the number of these variables matches the number of formulae. After solving the simultaneous equation, the resultant force distribution and stress distribution are obtained.

2.5. Average stiffness of laminates, stiffness of each layer and energy release rate

The stress distribution can be converted to the strain distribution by using the constitutive law of equations (7)–(9). The average strain in Layer 2 is calculated by numerically integrating the strain distribution. The average stiffness of laminate $\bar{Q}_{22}^{(c)}(c)$ can be obtained by equations (21)

$$\bar{Q}_{22}^{(c)}(c) = \frac{2L\sigma_{y0}Q_{22}^{(2)}(\theta)}{\Delta y \sum_{j=1}^{N}\{\sigma_y(j) + \sigma_y(j+1)\}}. \tag{21}$$

By substituting equation (21) into equation (1), the average stiffness of Layer 1 (90-degree Layer) $\bar{Q}_{22}^{(1)}(c)$ can be obtained. Because of the analogy described previously $\bar{Q}_{66}^{(1)}(c)$ can be obtained by the same procedure.

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The change of elastic energy due to the stiffness reduction of $[\theta/90]_s$ laminate results in the Mode I and Mode II energy release rate, G_{I} and G_{II}, by referring to equation (3).

$$G_{\mathrm{I}} = -\frac{b_1+b_2}{2b_1}\varepsilon_y^2 \frac{\partial \bar{Q}_{22}^{(c)}(c)}{\partial c} = -\frac{1}{2}\varepsilon_y^2 \frac{\partial \bar{Q}_{22}^{(1)}(c)}{\partial c},$$
$$G_{\mathrm{II}} = -\frac{b_1+b_2}{2b_1}\gamma_{xy}^2 \frac{\partial \bar{Q}_{66}^{(c)}(c)}{\partial c} = -\frac{1}{2}\gamma_{xy}^2 \frac{\partial \bar{Q}_{66}^{(1)}(c)}{\partial c}. \quad (22)$$

When calculating the equivalent stiffness reduction and strain of the layer in the relevant coordinate system, the energy release rate of the layer with growth of transverse lamina cracks can be obtained independently of other layers.

3. NUMERICAL SOLUTION

3.1. Non-dimensional crack density

As crack density c of the through-the-width lamina cracks is $1/(2L)$, the ratio, t_1c, of ply group thickness $t_1 = 2b_1$ to crack interval $2L$ in the Layer 1 (90-degree layer) is called the non-dimensional crack density. According to the similarity rule, the average stiffness and equivalent stiffness of laminate are described as a function of the non-dimensional crack density, t_1c.

$$\bar{Q}_{22}^{(1)}(c) = \bar{Q}_{22}^{(1)}(t_1c), \quad \bar{Q}_{22}^{(c)}(c) = \bar{Q}_{22}^{(c)}(t_1c),$$
$$\bar{Q}_{66}^{(1)}(c) = \bar{Q}_{66}^{(1)}(t_1c), \quad \bar{Q}_{66}^{(c)}(c) = \bar{Q}_{66}^{(c)}(t_1c). \quad (23)$$

In the present paper, all the results of analysis are given as a function of non-dimensional crack density, t_1c.

3.2. Convergence of numerical solutions

To check the convergence of the numerical solution, this calculation is done in various number of divisions while keeping constant the ratio of numbers of the vertical division to the horizontal division, M/N, of 1/2, 1 and 2. The relation between the number of divisions and numerical results is illustrated in Fig. 5 (ratio of material constants $E_{\mathrm{L}}/E_{\mathrm{T}} = 10$, non-dimensional crack density $t_1c = 0.5$, laminate configuration $[0_1/90_1]_s$). An accurate numerical solution is obtained when a small number of divisions is used, and there is only a small difference in the third digit even when the number of divisions (M, N) is $(16, 32)$. Moreover, as there is a linear relation between the inverse number of the product of the numbers of division M and N, $1/(M*N)$, and the numerical solutions, more accurate numerical solutions can be obtained by extrapolating numerical solutions to $1/(M*N) = 0$. The extrapolated value is adopted as the analytical solution in the present paper.

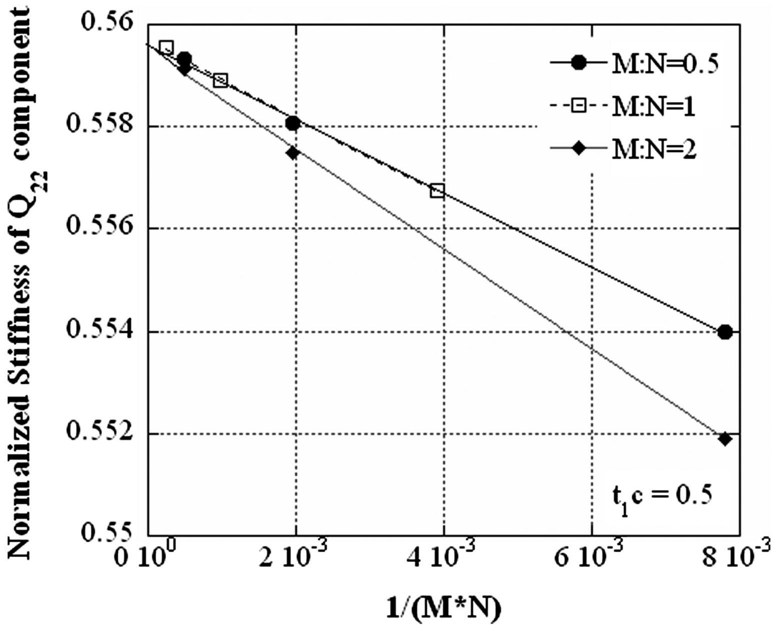

Figure 5. Effect of mesh division on numerical error.

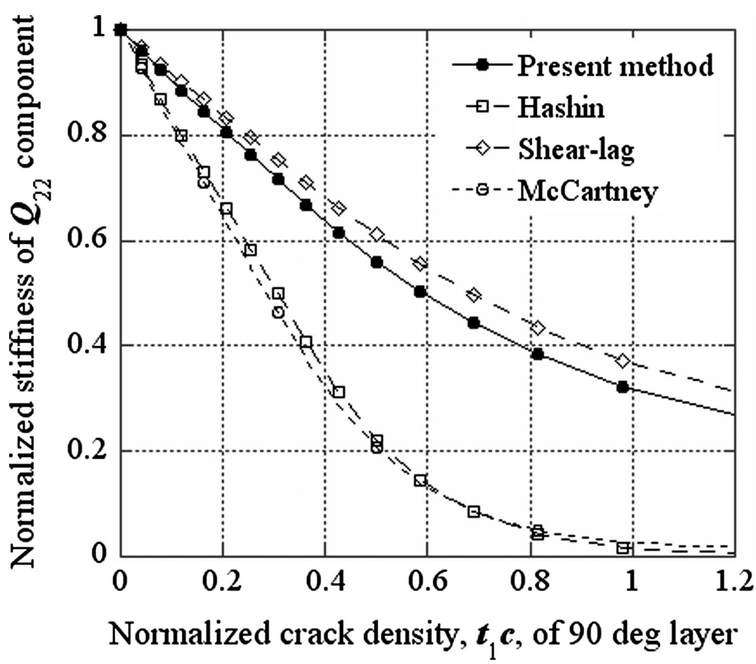

Figure 6. Comparison of analytical results.

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3.3. Comparison with other results

As for a cross-ply laminate $[0_1/90_1]_s$ ($E_L/E_T = 10$, $G_{LT}/E_T = 1/3$, $\nu_{LT} = 0.34$, $\nu_{TZ} = 0.4$), the relation between non-dimensional crack density and non-dimensional tensile stiffness calculated by the present method is shown in Fig. 6. Comparison with the shear-lag method, Hashin's solution and McCartney's solution is also shown in this figure. Though the present method gives less stiffness than the shear-lag method, it shows larger stiffness than Hashin's and McCartney's solutions because of the constraint of deformation in the thickness direction. Unlike Hashin's and McCartney's solutions, the present results do not give the lower bound of stiffness, because of constraint of deformation in the thickness direction. The assumption of the constraint is, nevertheless, applicable to the transverse lamina cracks in some critical layers in the multidirectionally reinforced laminates, especially in the initial stage of damage extension. The undamaged layers constrain the through-the-thickness deformation of damaged layers.

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4. PROPOSAL FOR INTERPOLATION FUNCTIONS FORMULA

Degradation of stiffness coefficients with extension of transverse lamina cracks is evaluated numerically by using the present method. The interpolation functions are proposed as a function of non-dimensional crack density, elastic moduli, ply thickness ratio and fiber orientation for transversally isotropic unidirectionally reinforced lamina. Poisson ratios, ν_{LT} and ν_{TZ}, are fixed to be 0.34 and 0.4 in the present paper, and the independent elastic constants are E_L, E_T and G_{LT} because of the feature of transversal isotropy as denoted by equation (24).

$$E_Z = E_T, \quad G_{TZ} = \frac{E_Z}{2(1+\nu_{TZ})} = \frac{E_T}{2(1+\nu_{TZ})}, \quad G_{ZL} = G_{LT}. \tag{24}$$

4.1. Scope of application

Interpolation functions for the laminates which consist of 0-degree, ±45-degree and 90-degree plies are proposed in the present paper. From this limitation, the angle with neighboring ply group is 90-degree or ±45-degree. Then only $\theta = 0$-degree and ±45-degree covers all the cases. The ranges of elastic moduli are $0.02 \leqslant E_T/E_L \leqslant 0.1$ and $1/3 \leqslant G_{LT}/E_T \leqslant 1/2$. The ratio of ply group thickness (Layer 1 to Layer 2) is in the range of $1/3 \leqslant t_1/t_2 \leqslant 3$.

4.2. Analytical expression of interpolation functions

The reduction of longitudinal stiffness can be described by the following two interpolation functions of non-dimensional crack density, t_1c. The error between the interpolation functions and the present numerical solutions is less than or equal to 1%.

$$\frac{\bar{Q}_{22}^{(1)}(t_1c)}{Q_{22}^{(1)}} = \begin{cases} k_0^{(22)} - k_1^{(22)} \times (t_1c) & 0 < t_1c \leqslant 0.4 \\ 1 - \dfrac{2}{\pi}\arctan\{\alpha_0^{(22)} + \alpha_1^{(22)} \times (t_1c)\} & 0.4 < t_1c. \end{cases} \tag{25}$$

Coefficients, $k_0^{(22)}$, $k_1^{(22)}$, $\alpha_0^{(22)}$ and $\alpha_1^{(22)}$ are the functions of elastic moduli ratio E_T/E_L and G_{LT}/E_T, fiber orientation angle θ and thickness ratio, t_1/t_2, of ply groups. G_{LT}/E_T affects the solution only in the case of $\theta = \pm 45$-degree, but the effect is small enough to be ignored in the present analysis. The coefficients in equation (25) can be expressed in first order functions of ratio of elastic moduli E_T/E_L according to the numerical solutions (superscript (22) indicates the case of longitudinal tensile).

$$k_0^{(22)}\left(\frac{E_T}{E_L}, \theta, \frac{t_1}{t_2}\right) = a_{00}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) + a_{01}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) \cdot \frac{E_T}{E_L}, \tag{26}$$

$$k_1^{(22)}\left(\frac{E_T}{E_L}, \theta, \frac{t_1}{t_2}\right) = a_{10}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) + a_{11}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) \cdot \frac{E_T}{E_L}, \tag{27}$$

$$\alpha_0^{(22)}\left(\frac{E_T}{E_L}, \theta, \frac{t_1}{t_2}\right) = d_{00}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) + d_{01}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) \cdot \frac{E_T}{E_L}, \tag{28}$$

$$\alpha_1^{(22)}\left(\frac{E_T}{E_L}, \theta, \frac{t_1}{t_2}\right) = d_{10}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) + d_{11}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) \cdot \frac{E_T}{E_L}. \tag{29}$$

In addition, the coefficients, $a_{ij}^{(22)}$ and $d_{ji}^{(22)}$, of equations (26)–(29) can be expressed by the first order function of ply group thickness ratio t_1/t_2.

$$a_{ij}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) = A_{ij}^{(22)}(\theta) + B_{ij}^{(22)}(\theta) \cdot \frac{t_1}{t_2} \quad (i, j = 0, 1), \tag{30}$$

$$d_{ij}^{(22)}\left(\theta, \frac{t_1}{t_2}\right) = C_{ij}^{(22)}(\theta) + D_{ij}^{(22)}(\theta) \cdot \frac{t_1}{t_2} \quad (i, j = 0, 1). \tag{31}$$

We can apply the same expressions of interpolation functions just as much to the in-plane shear problem as to the longitudinal tension. The error between the interpolation functions and the present numerical solutions is less than or equal to 1%.

$$\frac{\bar{Q}_{66}^{(1)}(t_1c)}{Q_{66}^{(1)}} = \begin{cases} k_0^{(66)} - k_1^{(66)}(t_1c) & 0 < t_1c \leqslant 0.5 \\ 1 - \dfrac{2}{\pi}\arctan\{\alpha_0^{(66)} + \alpha_1^{(66)}(t_1c)\} & 0.5 < t_1c. \end{cases} \tag{32}$$

The coefficients, $k_0^{(66)}$, $k_1^{(66)}$, $\alpha_0^{(66)}$ and $\alpha_1^{(66)}$ are given as a first order function of elastic modulus ratio E_T/E_L. (Superscript (66) means in-plane shear.)

$$k_0^{(66)}\left(\frac{E_T}{E_L}, \theta, \frac{t_1}{t_2}\right) = a_{00}^{(66)}\left(\theta, \frac{t_1}{t_2}\right) + a_{01}^{(66)}\left(\theta, \frac{t_1}{t_2}\right) \cdot \frac{E_T}{E_L}, \tag{33}$$

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$$k_1^{(66)}\left(\frac{E_\mathrm{T}}{E_\mathrm{L}}, \theta, \frac{t_1}{t_2}\right) = a_{10}^{(66)}\left(\theta, \frac{t_1}{t_2}\right) + a_{11}^{(66)}\left(\theta, \frac{t_1}{t_2}\right) \cdot \frac{E_\mathrm{T}}{E_\mathrm{L}}, \tag{34}$$

$$\alpha_0^{(66)}\left(\frac{E_\mathrm{T}}{E_\mathrm{L}}, \theta, \frac{t_1}{t_2}\right) = d_{00}^{(66)}\left(\theta, \frac{t_1}{t_2}\right) + d_{01}^{(66)}\left(\theta, \frac{t_1}{t_2}\right) \cdot \frac{E_\mathrm{T}}{E_\mathrm{L}}, \tag{35}$$

$$\alpha_1^{(66)}\left(\frac{E_\mathrm{T}}{E_\mathrm{L}}, \theta, \frac{t_1}{t_2}\right) = d_{10}^{(66)}\left(\theta, \frac{t_1}{t_2}\right) + d_{11}^{(66)}\left(\theta, \frac{t_1}{t_2}\right) \cdot \frac{E_\mathrm{T}}{E_\mathrm{L}}. \tag{36}$$

The coefficients, $a_{ij}^{(66)}$ and $d_{ij}^{(66)}$ are the linear function of the ratio of the ply group thickness.

$$a_j^{(66)}\left(\theta, \frac{t_1}{t_2}\right) = A_j^{(66)}(\theta) + B_j^{(66)}(\theta) \cdot \frac{t_1}{t_2} \quad (j = 0, 1), \tag{37}$$

$$d_{ij}^{(66)}\left(\theta, \frac{t_1}{t_2}\right) = C_{ij}^{(66)}(\theta) + D_{ij}^{(66)}(\theta) \cdot \frac{t_1}{t_2} \quad (i, j = 0, 1). \tag{38}$$

Because we assume linear elasticity in the present study, the material non-linearity of shear deformation in $\theta = 0$-degree payer is not considered. So only the $\theta = \pm 45$-degree case is calculated in the present analysis.

The coefficients of equations (30) and (31) and equations (37) and (38) are shown in Table 1.

4.3. Calculation of energy release rate

The energy release rate given by equation (22) can be calculated analytically from the interpolation functions of stiffness reduction obtained thus far. So as for Mode I, the following equation is obtained.

$$G_\mathrm{I} = \begin{cases} \dfrac{1}{2} k_1^{(22)} t_1 Q_{22}^{(1)} \cdot \varepsilon_y^2 & 0 < t_1 c \leqslant 0.4 \\ \dfrac{1}{\pi} t_1 Q_{22}^{(1)} \cdot \varepsilon_y^2 \cdot \dfrac{\alpha_1^{(22)}}{1 + \{\alpha_0^{(22)} + \alpha_1^{(22)} \times (t_1 c)\}^2} & 0.4 < t_1 c. \end{cases} \tag{39}$$

In the same way, the stiffness reduction in in-plane shear stiffness gives the Mode II energy release rate.

$$G_\mathrm{II} = \begin{cases} \dfrac{1}{2} k_1^{(66)} t_1 Q_{66}^{(1)} \cdot \gamma_{xy}^2 & 0 < t_1 c \leqslant 0.5 \\ \dfrac{1}{\pi} t_1 Q_{66}^{(1)} \cdot \gamma_{xy}^2 \cdot \dfrac{\alpha_1^{(66)}}{1 + \{\alpha_0^{(66)} + \alpha_1^{(66)} \times (t_1 c)\}^2} & 0.5 < t_1 c. \end{cases} \tag{40}$$

The energy release rate in the initial stage of transverse lamina cracks ($t_1 c \leqslant 0.4$) is constant independent of crack density when the strains ε_y and γ_{yz} are constant. When the crack density increases and the strain is kept constant, the energy release rate changes and decreases. In other words, the damage will progress stably. The energy release rate is in proportion to ply group thickness t_1, and as the thickness

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Table 1.
Coefficients of interpolation functions

A. Tensile deformation (Mode I)

	$k_0^{(22)}$				$k_1^{(22)}$			
	$a_{00}^{(22)}$		$a_{01}^{(22)}$		$a_{10}^{(22)}$		$a_{11}^{(22)}$	
θ	$A_{00}^{(22)}$	$B_{00}^{(22)}$	$A_{01}^{(22)}$	$B_{01}^{(22)}$	$A_{01}^{(22)}$	$B_{10}^{(22)}$	$A_{11}^{(22)}$	$B_{11}^{(22)}$
0°			−0.002	−0.023	0.889	0.000	0.050	0.063
±45°	0.997	0.000	−0.011	−0.051		0.002	0.070	0.150
	$\alpha_0^{(22)}$				$\alpha_1^{(22)}$			
	$d_{00}^{(22)}$		$d_{01}^{(22)}$		$d_{10}^{(22)}$		$d_{11}^{(22)}$	
θ	$C_{00}^{(22)}$	$D_{00}^{(22)}$	$C_{01}^{(22)}$	$D_{01}^{(22)}$	$C_{10}^{(22)}$	$D_{10}^{(22)}$	$C_{11}^{(22)}$	$D_{11}^{(22)}$
0°		0.000	0.003	0.090			0.115	0.000
±45°	−0.180	0.003	0.019	0.233	1.997	0.000	0.112	0.002

B. Shear deformation (Mode II)

	$k_0^{(66)}$				$k_1^{(66)}$			
	$a_{00}^{(66)}$		$a_{01}^{(66)}$		$a_{10}^{(66)}$		$a_{11}^{(66)}$	
θ	$A_{00}^{(66)}$	$B_{00}^{(66)}$	$A_{01}^{(66)}$	$B_{01}^{(66)}$	$A_{10}^{(66)}$	$B_{10}^{(66)}$	$A_{11}^{(66)}$	$B_{11}^{(66)}$
±45°	0.998	0.000	−0.002	−0.023	0.034	0.000	0.004	0.061
	$\alpha_0^{(66)}$				$\alpha_1^{(66)}$			
	$d_{00}^{(66)}$		$d_{01}^{(66)}$		$d_{10}^{(66)}$		$d_{11}^{(66)}$	
θ	$C_{00}^{(66)}$	$D_{00}^{(66)}$	$C_{01}^{(66)}$	$D_{01}^{(66)}$	$C_{10}^{(66)}$	$D_{10}^{(66)}$	$C_{11}^{(66)}$	$D_{11}^{(66)}$
±45°	−0.192	0.000	0.004	0.114	1.199	0.000	0.000	−0.001

becomes thinner the energy release rate becomes smaller at the same state of the applied strain.

4.4. Effect of transverse cracks on elastic modulus

Though the previous discussion is about the elastic stiffness, $\nu_{LT}\nu_{TL}$ is much smaller than 1, so the engineering elastic constant, E_T, of normal direction to reinforcing fiber is satisfied with following relation.

$$\frac{\bar{Q}_{22}^{(1)}(t_1c)}{Q_{22}^{(1)}} = \frac{E_T(t_1c)(1-\nu_{LT}\nu_{TL})}{E_T\{1-\nu_{LT}\nu_{TL}(t_1c)\}} = \frac{E_T(t_1c)}{E_T}\cdot\left[1-\nu_{LT}^2\frac{E_T}{E_L}\left\{1-\frac{E_T(t_1c)}{E_T}\right\}\right]. \quad (41)$$

ν_{LT} is about 0.34 and $E_{\mathrm{T}}/E_{\mathrm{L}}$ is assumed to be less than or equal to 0.1 in the present analysis, so the following equation is acceptable with range of error less than 1%.

$$\frac{\bar{Q}_{22}^{(1)}(t_1 c)}{Q_{22}^{(1)}} = \frac{E_{\mathrm{T}}(t_1 c)}{E_{\mathrm{T}}}. \tag{42}$$

This means that the same interpolation functions are applicable to the engineering tensile and shear moduli.

$$\frac{E_{\mathrm{T}}(t_1 c)}{E_{\mathrm{T}}} = \begin{cases} k_{0_0}^{(22)} - k_1^{(22)} \times (t_1 c) & 0 < t_1 c \leqslant 0.4 \\ 1 - \dfrac{2}{\pi} \arctan\{\alpha_0^{(22)} + \alpha_1^{(22)} \times (t_1 c)\} & 0.4 < t_1 c, \end{cases} \tag{43}$$

$$\frac{G_{\mathrm{LT}}(t_1 c)}{G_{\mathrm{LT}}} = \begin{cases} k_{0_0}^{(66)} - k_1^{(66)}(t_1 c) & 0 < t_1 c \leqslant 0.5 \\ 1 - \dfrac{2}{\pi} \arctan\{\alpha_0^{(66)} + \alpha_1^{(66)}(t_1 c)\} & 0.5 < t_1 c. \end{cases} \tag{44}$$

5. PROPOSAL FOR DAMAGE TOLERANT DESIGN

Behavior of initiation and progress of transverse lamina cracks in a composite laminate with unidirectionally reinforced ply groups whose fiber orientation angles are 0-degree, ±45-degree and 90-degree is discussed here. A unidirectional reinforced composite shows significant material non-linearity under shear deformation, but composite materials used in practice, such as a quasi-isotropic laminate, are designed not to show material non-linearity under designed applied load. Shear strain in a ply group is limited and Mode I deformation is supposed to be dominant.

Damage initiates and progresses when both failure criteria on stress and energy states are satisfied [12]. The stress criterion does not depend on the thickness of each ply group. On the other hand, the critical strain of the energy criterion becomes larger as the thickness of each ply group becomes thinner. So, when the thickness of a ply group is thinner than some limiting value, energy criterion governs the initiation and progress of damage. This critical thickness of a ply group is given as the thickness that has the same critical strain in both stress criterion and energy criterion. If it is assumed that Mode I is dominant, the limiting value of thickness of ply group t (without suffix) is roughly calculated by equation (45), using the tensile strength, F_{T}, in the 90-degree direction and Mode I fracture toughness , G_{Ic}.

$$t \leqslant \frac{2 E_{\mathrm{T}} G_{\mathrm{Ic}}}{k_1^{(22)} F_{\mathrm{T}}^2}. \tag{45}$$

We apply the mixed Mode I and II criterion proposed by Reeder [13].

$$\left(\frac{G_{\mathrm{I}}}{G_{\mathrm{Ic}}}\right)^m + \left(\frac{G_{\mathrm{II}}}{G_{\mathrm{IIc}}}\right)^n = 1. \tag{46}$$

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In-plane strains ε_y and γ_{xy} defined in the material coordination system are discussed here (suffix x and y means fiber orientation and normal orientation, respectively). However, in-plane strain here means only the elastic component without residual strain caused by molding or moisture. As for equations (39) and (40), the initiation of transverse lamina crack initiation is determined by the following equation.

$$\left\{\frac{k_1^{(22)} t E_{\mathrm{T}}}{2G_{\mathrm{Ic}}}\varepsilon_y^2\right\}^m + \left\{\frac{k_1^{(66)} t G_{\mathrm{LT}}}{2G_{\mathrm{IIc}}}\gamma_{xy}^2\right\}^n = 1. \tag{47}$$

Then if we assume $m = n = 1$, equation (48) is obtained.

$$\frac{k_1^{(22)} t E_{\mathrm{T}}}{2G_{\mathrm{Ic}}}\varepsilon_y^2 + \frac{k_1^{(66)} t G_{\mathrm{LT}}}{2G_{\mathrm{IIc}}}\gamma_{xy}^2 = 1. \tag{48}$$

When the strain is satisfied with this failure criterion, the failure progresses unsteadily until crack density $c = 0.4/t$ under the fixed strain condition. After the initial state, the behavior of damage progress is given in the following formula.

$$\frac{t E_{\mathrm{T}}}{\pi G_{\mathrm{Ic}}} \cdot \frac{\alpha_1^{(22)}}{1 + \{\alpha_0^{(22)} + \alpha_1^{(22)} \times (tc)\}^2} \cdot \varepsilon_y^2 + \frac{t G_{\mathrm{LT}}}{\pi G_{\mathrm{IIc}}} \cdot \frac{\alpha_1^{(66)}}{1 + \{\alpha_0^{(66)} + \alpha_1^{(66)} \times (tc)\}^2} \cdot \gamma_{xy}^2 = 1. \tag{49}$$

$\alpha_1^{(22)}$ and $\alpha_1^{(66)}$ do not depend on elastic moduli or ratio of ply group thickness, and they are about 2.0 and 1.2, respectively. The terms of $\alpha_0^{(22)}$ and $\alpha_0^{(66)}$ are negligible because the values are less than or equal to 0.2 after the extensive progress of transverse lamina cracks. The following equation is applicable under the progressively damaged laminate.

$$\frac{t E_{\mathrm{T}}}{2\pi G_{\mathrm{Ic}}} \cdot \left(\frac{\varepsilon_y}{tc}\right)^2 + \frac{t G_{\mathrm{LT}}}{1.2\pi G_{\mathrm{IIc}}} \cdot \left(\frac{\gamma_{xy}}{tc}\right)^2 = 1. \tag{50}$$

Nondimensional crack density is calculated as a function of the stress state.

$$tc = \sqrt{\frac{t E_{\mathrm{T}}}{2\pi G_{\mathrm{Ic}}} \cdot \varepsilon_y^2 + \frac{t G_{\mathrm{LT}}}{1.2\pi G_{\mathrm{IIc}}} \cdot \gamma_{xy}^2}. \tag{51}$$

The above equation shows that non-dimensional crack density increases in proportion to strain and the increasing rate is affected by the square root of the ply group thickness – the fracture toughness value quotient. As non-dimensional crack density is given as a function of strain, the reduction of elastic moduli is calculated as a function of strains by applying equation (51) and equations (43) and (44).

6. CONCLUSION

A method for analyzing growth of transverse lamina cracks in multi-directionally reinforced laminates with unidirectional laminae whose fiber orientation angles are 0-degree, ±45-degree and 90-degree is proposed here. A method for calculating

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stress distribution in the $[\theta_m/90_n]_s$ laminate, which is a basic unit of the model in this paper, is also proposed. This method gives a numerical stress distribution using resultant force differential method. Almost half the number of freedom of equations gives the same results of stress distribution, when compared with the method previously proposed by the authors. And accurate interpolation expressions are also proposed by carrying out many cases of calculations with various fiber orientation angle, elastic modulus and ply group thickness.

A simplified calculation method for evaluation of damage tolerant design is proposed in this the present paper based on the interpolation functions.

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